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On theta characteristics of a compact Riemann surface

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Abstract

Let σ be a nontrivial automorphism of a compact connected Riemann surface X of genus at least two. Assume that σ fixes each of the theta characteristics of X. We prove that X is hyperelliptic, and σ is the unique hyperelliptic involution of X.

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1. Introduction

Let X be a compact connected Riemann surface with $g := \text{genus}(X) > 1$. Theta characteristics on X are the square-roots of the holomorphic cotangent bundle of X. Since the degree of the holomorphic cotangent bundle of X is an even integer, there are theta characteristics on X. In fact, there are exactly 4^g of them. Theta characteristics are very important objects associated to a Riemann surface; see [2,4].

Any holomorphic automorphism on X acts on the set of all theta characteristics on X. The action of any automorphism σ of X sends a theta characteristic L to σ^*L. Our aim here is to investigate the action of a holomorphic automorphism of X on its theta characteristics.

We prove the following theorem (see Theorem 2.1):

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Theorem 1.1. *Let σ be a holomorphic automorphism of X that fixes pointwise all the theta characteristics on X. Assume that $\sigma \neq \mathrm{Id}_X$. Then X is a hyperelliptic Riemann surface, and furthermore, σ is the unique hyperelliptic involution of X.*

We recall that a hyperelliptic Riemann surface is a double cover of $\mathbb{CP}^1$, and a hyperelliptic involution is an automorphism of order two for which the quotient is isomorphic to $\mathbb{CP}^1$. We note that any hyperelliptic involution fixes all the theta characteristics pointwise (see Remark 2.2).

Theorem 1.1 can be strengthened in the following way: If σ is a nontrivial holomorphic automorphism of X that fixes pointwise all the even theta characteristics on X, then σ must be a hyperelliptic involution. (See Proposition 2.3.)

We note that Theorem 1.1 remains valid for elliptic curves if we restrict ourselves to automorphisms that fix some point; see Remark 2.4.

Proof of Theorem 1.1 will be carried out in two parts. First we shall prove that $\sigma^2 = \mathrm{Id}_X$. Then it will be shown that the quotient of X by σ is isomorphic to $\mathbb{CP}^1$.

2. Automorphisms and the theta characteristics

Let X be a compact connected Riemann surface of genus g, with $g \geqslant 2$. The holomorphic cotangent bundle of X will be denoted by K_X.

A *theta characteristic* on X is a holomorphic line bundle L of degree $g-1$ over X such that the tensor product $L \otimes L$ is holomorphically isomorphic to the cotangent line bundle K_X. Let $\mathcal{S}(X)$ denote the space of all isomorphism classes of theta characteristics on X. We note that $\mathcal{S}(X)$ is an affine space for the group of all order two holomorphic line bundles on X. The action on $\mathcal{S}(X)$ of any holomorphic line bundle ξ of order two sends any theta characteristic L to $L \otimes \xi$.

We will denote by $J(X)_2$ the group of all isomorphism classes of holomorphic line bundles on X of order two. There is a natural isomorphism of groups

$$\mu : J(X)_2 \longrightarrow H^1(X; \mathbb{Z}/2\mathbb{Z}) \tag{1}$$

which can be constructed as follows. Given a holomorphic line bundle L on X of order two, fix a holomorphic isomorphism

$$\alpha : L^{\otimes 2} \longrightarrow X \times \mathbb{C},$$

where $X \times \mathbb{C}$ is the trivial line bundle over X. Let

$$\beta : L \longrightarrow L^{\otimes 2}$$

be the (nonlinear) holomorphic map between the total spaces of line bundles defined by $w \longmapsto w \otimes w$. Let

$$p : (\alpha \circ \beta)^{-1}\big(X \times \{1\}\big) \longrightarrow X$$

be the restriction of the natural projection $L \longrightarrow X$. This map p is evidently a unramified covering of degree two. Therefore, the covering p gives an element

$$\widetilde{p} \in \mathrm{Hom}\big(\pi_1(X), \mathbb{Z}/2\mathbb{Z}\big) = \mathrm{Hom}\big(H_1(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}\big) = H^1(X; \mathbb{Z}/2\mathbb{Z}).$$

It is easy to see that the isomorphism class of the covering p is independent of the choice of the isomorphism α. With the above notation, the homomorphism μ in (1) is defined by

$$\mu(L) = \widetilde{p}.$$

Using the isomorphism μ, the set $\mathcal{S}(X)$ of theta characteristics on X is an affine space for the group $H^1(X;\mathbb{Z}/2\mathbb{Z})$. In particular, there are exactly 4^g distinct theta characteristics on X. Let

$$\phi : \mathcal{S}(X) \times H^1(X;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(X;\mathbb{Z}/2\mathbb{Z}) \tag{2}$$

be the action of $H^1(X;\mathbb{Z}/2\mathbb{Z})$ on $\mathcal{S}(X)$.

We recall that a compact connected Riemann surface Y is called *hyperelliptic* if it admits a nonconstant holomorphic map, of degree two, to the complex projective line $\mathbb{CP}^1$. A holomorphic automorphism τ of Y is called a *hyperelliptic involution* if $\tau \circ \tau = \mathrm{Id}_Y$ with $\mathrm{genus}(Y/\langle\tau\rangle) = 0$, where $Y/\langle\tau\rangle$ is the quotient of Y by τ. If Y is hyperelliptic, and $\mathrm{genus}(Y) \geqslant 2$, then Y admits exactly one hyperelliptic involution [3, p. 108, Corollary 2].

Let

$$\sigma : X \longrightarrow X \tag{3}$$

be a holomorphic automorphism of the Riemann surface X. The automorphism σ acts on the group of all holomorphic line bundles on X by sending any L to its pull back σ^*L. This action clearly takes the holomorphic cotangent bundle K_X to itself. Therefore, σ acts on the set of all theta characteristics on X.

Theorem 2.1. *Let σ be a holomorphic automorphism of X that acts trivially on the set of all theta characteristics on X. Assume that $\sigma \neq \mathrm{Id}_X$. Then σ is an involution, and the quotient $X/\langle\sigma\rangle$ is isomorphic to $\mathbb{CP}^1$. In other words, X is hyperelliptic, and σ is the unique hyperelliptic involution of X.*

Proof. The automorphism σ in (3) acts on both $J(X)_2$ and $H^1(X;\mathbb{Z}/2\mathbb{Z})$, and the isomorphism μ in (1) intertwines these two actions. Also, the map ϕ in (2) evidently commutes with the actions of σ, with σ acting diagonally on $\mathcal{S}(X) \times H^1(X;\mathbb{Z}/2\mathbb{Z})$. Therefore, the given condition that σ acts trivially on the set of all theta characteristics on X immediately implies that σ acts trivially on

$$H^1(X;\mathbb{Z}/2\mathbb{Z}) = H^1(X;\mathbb{Z}/2\mathbb{Z})^* = H_1(X;\mathbb{Z}/2\mathbb{Z}).$$

The above isomorphism between $H^1(X;\mathbb{Z}/2\mathbb{Z})$ and $H^1(X;\mathbb{Z}/2\mathbb{Z})^*$ is obtained from the cup product.

Since $g \geqslant 2$, the holomorphic automorphism group of X is finite; a theorem of Hurwitz says that the cardinality of $\mathrm{Aut}(X)$ is at most $84(g-1)$, where $\mathrm{Aut}(X)$ is the group of all holomorphic automorphisms of X (see [3, p. 258, Theorem]). In particular, the automorphism σ is of finite order. Let

$$T : H_1(X;\mathbb{Z}) \longrightarrow H_1(X;\mathbb{Z}) \tag{4}$$

be the isomorphism given by the automorphism σ.

If we fix a symplectic basis of $H_1(X;\mathbb{Z})$, then T is given by an element in $\mathrm{Sp}(2g,\mathbb{Z})$. Let

$$T_2 : H_1(X;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(X;\mathbb{Z}/2\mathbb{Z})$$

be the automorphism induced by T. This automorphism T_2 evidently coincides with the one given by σ. We noted above that σ acts trivially on $H_1(X;\mathbb{Z}/2\mathbb{Z})$. Therefore, $T_2 = \mathrm{Id}_{H_1(X;\mathbb{Z}/2\mathbb{Z})}$. From this it follows that

$$T^2 = \mathrm{Id}_{H_1(X;\mathbb{Z})} \tag{5}$$

(see [3, p. 292, Lemma]).

The natural homomorphism $\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(H_1(X;\mathbb{Z}))$ is injective [3, p. 287, Theorem]. Consequently, from (5) we conclude that

$$\sigma \circ \sigma = \mathrm{Id}_X. \tag{6}$$

(See also [3, p. 293, Theorem].)

Any holomorphic automorphism of X preserves the unique hyperbolic metric on X of constant curvature -1. Therefore, from Proposition 3.1, which is stated and proved in the next section, we know that the quotient surface $X/\langle\sigma\rangle$ is isomorphic to $\mathbb{CP}^1$. We already noted that a compact connected Riemann surface of genus at least two admits at most one hyperelliptic involution. This completes the proof of the theorem. □

Remark 2.2. Let X be a hyperelliptic Riemann surface. Let σ be the hyperelliptic involution of X. Then σ fixes each of the theta characteristics on X; see [1, p. 288, 32(i)].

We recall that the set $\mathcal{S}(X)$ of theta characteristics of X can be partitioned into two subsets $\mathcal{S}_i(X)$, $i = 0, 1$, where $\mathcal{S}_i(X)$ consists of all those line bundles L for which $\dim_{\mathbb{C}} H^0(X; L) \equiv i$ mod 2. The elements of $\mathcal{S}_0(X)$ are called the *even theta characteristics*, while the elements of $\mathcal{S}_1(X)$ are called the *odd theta characteristics*. It is known that there are exactly $2^{g-1}(2^g + 1)$ elements in $\mathcal{S}_0(X)$ and $2^{g-1}(2^g - 1)$ elements in $\mathcal{S}_1(X)$ [4, p. 190, § 4]. Clearly the action of any automorphism of X on $\mathcal{S}(X)$ preserves both $\mathcal{S}_0(X)$ and $\mathcal{S}_1(X)$.

The following proposition, which is proved using Theorem 2.1, gives a stronger version Theorem 2.1.

Proposition 2.3. *Let σ be a holomorphic automorphism of X. Assume that $\sigma \neq \mathrm{Id}_X$. If σ fixes pointwise all the even theta characteristics on X, then X is hyperelliptic, and σ is the unique hyperelliptic involution of X.*

Proof. Assume that σ fixes pointwise all the even theta characteristics $\mathcal{S}_0(X)$ on X. Fix an element $L_0 \in \mathcal{S}_0(X)$.

We noted earlier that the set of all theta characteristics $\mathcal{S}(X)$ is an affine space for the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^1(X;\mathbb{Z}/2\mathbb{Z})$. Taking L_0 as the origin, we identity $\mathcal{S}(X)$ with $H^1(X;\mathbb{Z}/2\mathbb{Z})$ in a $H^1(X;\mathbb{Z}/2\mathbb{Z})$-equivariant way.

Let

$$\Gamma_0 \subset H^1(X;\mathbb{Z}/2\mathbb{Z})$$

be the subgroup generated by the subset $\mathcal{S}_0(X)$ using the above $H^1(X;\mathbb{Z}/2\mathbb{Z})$-equivariant isomorphism of $\mathcal{S}(X)$ with $H^1(X;\mathbb{Z}/2\mathbb{Z})$. If σ fixes $\mathcal{S}_0(X)$ pointwise, then it follows immediately that the action of σ on $H^1(X;\mathbb{Z}/2\mathbb{Z})$ fixes Γ_0 pointwise.

The order $\#\Gamma_0$ of the subgroup Γ_0 is submultiple of the order $\#H^1(X;\mathbb{Z}/2\mathbb{Z})$. On the other hand,

$$2 \cdot \#\Gamma_0 = 2^g(2^g + 1) > 4^g = \#H^1(X;\mathbb{Z}/2\mathbb{Z}).$$

Therefore, $\Gamma_0 = H^1(X;\mathbb{Z}/2\mathbb{Z})$. Consequently, if σ fixes $\mathcal{S}_0(X)$ pointwise, then it actually fixes $\mathcal{S}(X)$ pointwise. Now the proof of the proposition is completed using Theorem 2.1. □

Remark 2.4. Let Z be a compact connected Riemann surface of genus one. Fix a point $z_0 \in Z$. There is a unique complex (abelian) Lie group structure on Z with z_0 as the identity element.

This group is identified with $\mathrm{Pic}^0(Z)$ by sending any $z \in Z$ to the holomorphic line bundle over X defined by the divisor $z - z_0$. Any holomorphic automorphism of Z that fixes z_0 is actually an automorphism of the Lie group.

Let

$$\tau : Z \longrightarrow Z$$

be a holomorphic automorphism of the group that fixes pointwise the order two points of Z. From [3, p. 292, Lemma] it follows that $\tau \circ \tau = \mathrm{Id}_Z$.

Since $\tau \circ \tau = \mathrm{Id}_Z$, the corresponding automorphism of the Lie algebra of Z is either the identity map, or it is multiplication by -1. Any holomorphic automorphism of the Lie group Z is uniquely determined by the corresponding automorphism of the Lie algebra of Z. Therefore, we now conclude that either $\tau = \mathrm{Id}_Z$ or τ is the inversion map defined by $z \longmapsto -z$. The inversion map is evidently a hyperelliptic involution of Z.

In the next section we will prove Proposition 3.1.

3. Involutions acting trivially on homology mod 2

We consider in this section all the orientation preserving involutions of compact hyperbolic surfaces that act trivially on homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. By a theorem of Nielsen [5] such an involution is homotopic to an isometry of some hyperbolic metric on the surface.

We make the following convention throughout. If C is an (unoriented) curve on a surface S, then $[C]$ is the corresponding class in $H_1(S;\mathbb{Z}/2\mathbb{Z})$. Further if $[C]$ and $[C']$ are such homology classes, then $[C]\cdot[C']$ denotes their mod 2 intersection number.

Proposition 3.1. *Let $\tau : F \longrightarrow F$ be an orientation preserving isometric nontrivial involution of a compact hyperbolic surface F such that the induced homomorphism*

$$\tau_* : H_1(F;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(F;\mathbb{Z}/2\mathbb{Z})$$

is the identity map. Then the quotient S of F by the action of τ is a sphere.

Proof. Note that as τ is a hyperbolic isometry, the quotient $S := F/\langle\tau\rangle$ is a surface, and furthermore, the quotient map

$$\pi : F \longrightarrow S \tag{7}$$

is a branched covering whose branch points are the images of the fixed points of τ.

We first assume that the automorphism τ has a fixed point. Let $p_1, \ldots, p_d$ be the fixed points of τ. Let $q_i := \pi(p_i)$ be the image under the quotient map in (7).

Set $F' = F - \{p_1, \ldots, p_d\}$ and $S' = S - \{q_1, \ldots, q_d\}$. Then the restriction of π gives a 2-fold covering

$$\pi' := \pi|_{F'} : F' \longrightarrow S'. \tag{8}$$

This corresponds to a surjective homomorphism

$$\widehat{\varphi} : \pi_1(S') \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

This homomorphism $\widehat{\varphi}$ in turn factors through a surjective homomorphism

$$\varphi : H_1(S';\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}. \tag{9}$$

Let the genus of the surface S be m. Let $a_1, b_1, \dots, a_m, b_m$ be a standard system of curves for S. This means that $\{a_1, b_1, \dots, a_m, b_m\}$ is a collection of smooth simple curves in S such that a_i and b_i intersect in a single point transversally for each i, and all other pairs of curves in the system are disjoint, and furthermore, the complement $S \setminus \{a_1, b_1, \dots, a_m, b_m\}$ is connected (hence it is an open disk). We assume that each of the curves is contained in S'. Then the complement $S' \setminus \{a_1, b_1, \dots, a_m, b_m\}$ is a disk with punctures $q_1, \dots, q_d$. Let μ_i be the boundary of a small disk neighborhood of q_i in S.

Note that the collection of elements $[a_i]$, $[b_i]$ and $[\mu_i]$ represent a system of generators for $H_1(S'; \mathbb{Z}/2\mathbb{Z})$. Hence the homomorphism φ in (9) is determined by its values on these elements. Further, note that $\varphi(\mu_i) = 1$ for all i by elementary covering space theory.

We need the following lemma for the proof of the proposition.

Lemma 3.2. *The standard system of curves can be chosen so that $\varphi([a_i]) = \varphi([b_i]) = 0$ for all i.*

Proof. Suppose $\varphi([a_j]) = 1$. Let α be a smooth arc with endpoints on a_j and μ_1 whose interior is disjoint from all the curves a_i, b_i and μ_i for all i (recall that we have assumed that there is at least one fixed point). Let a'_j be the component of the boundary of a regular neighborhood of $a_j \cup \alpha \cup \mu_1$ that intersects the boundary of regular neighborhoods of each of the curves a_j, α and μ_j. Geometrically, this is the curve obtained by sliding a_j over μ_1 along the arc α.

Observe that in $H_1(S'; \mathbb{Z}/2\mathbb{Z})$, we have the relation $[a'_j] = [a_j] + [\mu_1]$. Hence $\varphi([a'_j]) = 0$. Further note that replacing a_j by a'_j gives a standard system of curves for S. Note that the other curves of the standard system have not been affected.

We modify the standard system in a similar way for each curve a_k or b_k with image 1 under φ. The resulting system of curves is as required. □

Continuing with the proof of the proposition, we assume henceforth that the system of curves has been chosen as above. By elementary covering space theory, it follows that the inverse image of each of the curves a_i and b_i, $1 \leqslant i \leqslant m$, consists of two components, each a simple closed curve mapping homeomorphically under the projection π' in (8); recall that all the curves a_i, b_i are contained in S'.

Suppose now that the genus m of S is at least one. Let x be the point of intersection of a_1 and b_1 and let y be one point $\pi^{-1}(x)$. Let A_1 and B_1 be the components containing y of $\pi^{-1}(a_1)$ and $\pi^{-1}(b_1)$ respectively. Then A_1 and B_1 intersect transversely in one point, hence the mod 2 intersection number $[A_1] \cdot [B_1]$ is 1. On the other hand, the curve $[\tau(A_1)]$ is the other component of $\pi^{-1}(a_1)$, which is disjoint from B_1. This implies that $\tau_*([A_1]) \cdot [B_1]$ is zero.

It follows that $\tau_*([A_1]) \neq [A_1]$ in $H_1(F; \mathbb{Z}/2\mathbb{Z})$, contradicting the hypothesis that τ_* induces the identity on $H_1(F; \mathbb{Z}/2\mathbb{Z})$. Thus, if τ has a fixed point, then S has genus zero, i.e., S is a sphere. This completes the proof of the proposition under the assumption that τ has a fixed point.

Now assume that τ does not have any fixed points. Therefore, $S = S'$, and $\pi = \pi' : F \longrightarrow S$ is a unramified covering.

We note that S is not a sphere as F is connected, and S is not a torus as its cover F has genus at least two. Hence F must have genus at least two.

As before, we have the surjective homomorphism $\varphi : H_1(S; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ constructed in (9). We choose a standard system of curves a_i, b_i, $1 \leqslant i \leqslant m$, for S as before. Recall that we have $m > 1$.

Lemma 3.3. *The standard system of curves can be chosen so that*

(1) $\varphi([a_1]) = 1$,
(2) $\varphi([a_i]) = 0$ *for* $i \geqslant 2$, *and*
(3) $\varphi([b_i]) = 0$ *for all* i.

Proof. We first ensure that the condition on the curves b_i, $1 \leqslant i \leqslant m$, is satisfied. Suppose $\varphi([b_j]) = 1$ for some j. If $\varphi([a_j]) = 0$ we note that we can interchange the (unoriented) curves a_j and b_j to get a new standard system of curves so that $\varphi([b_j]) = 0$. On the other hand, if $\varphi([a_j]) = 1$, we can replace b_j by its image under a Dehn twist about a_j to get a new standard collection of curves with $\varphi([b_j]) = 0$. Thus, we can ensure that $\varphi([b_i]) = 0$ for each i.

We now view the surface S as the boundary of a handlebody H with the curves a_i the cores of 1-handles and the curves b_i the meridians (i.e., curves bounding disks in H). By the condition on the curves b_j, the map φ factors through a map

$$\varphi' : H_1(H; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

We shall use homeomorphisms of the handlebody H, restricted to its boundary S, to modify the standard system of curves. Note that a modification by such a homeomorphism automatically preserves the condition that $\varphi([b_i]) = 0$ for all i as this condition holds for all homomorphisms factoring through $H_1(H; \mathbb{Z}/2\mathbb{Z})$. Thus, it suffices to show that such modifications can ensure that the first two conditions hold.

Firstly, as φ is a surjection, for some j we have $\varphi([a_j]) = 1$. Using a homeomorphism that interchanges 1-handles, we can ensure that $\varphi([a_1]) = 1$. Now if for some $j > 1$ we have $\varphi([a_j]) = 1$, we handle slide the corresponding handle over the 1-handle corresponding to a_1. After this modification we obtain $\varphi([a_j]) = 0$. By performing such modifications for all j with $\varphi([a_j]) = 1$, we obtain a standard system of curves satisfying all the desired conditions. This completes the proof of the lemma. □

We now complete the proof of the proposition as in the case with fixed points. Recall that $m > 2$. As $\varphi([a_2]) = 0$ and $\varphi([b_2]) = 0$, we can choose components A_2 and B_2 of the inverse images of these curves intersecting transversally in one point, with $\tau(A_2)$ disjoint from B_2. As before, we deduce that $\tau_*([A_2]) \neq [A_2]$, a contradiction.

Thus, it is impossible that τ acts without fixed points, and in the case where τ has fixed points the quotient is a sphere. This completes the proof of the proposition. □

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